

ANALYTIC CONTINUATION OF RESOLVENT KERNELS ON NONCOMPACT SYMMETRIC SPACES

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ABSTRACT. Let $X = G/K$ be a symmetric space of noncompact type and let Δ be the Laplacian associated with a G -invariant metric on X . We show that the resolvent kernel of Δ admits a holomorphic extension to a Riemann surface depending on the rank of the symmetric space. This Riemann surface is a branched cover of the complex plane with a certain part of the real axis removed. It has a branching point at the bottom of the spectrum of Δ . It is further shown that this branching point is quadratic if the rank of X is odd, and is logarithmic otherwise. In case G has only one conjugacy class of Cartan subalgebras the resolvent kernel extends to a holomorphic function on a branched cover of \mathbb{C} with the only branching point being the bottom of the spectrum.

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1. INTRODUCTION

If (M, g) is a complete Riemannian manifold the metric Laplace operator Δ_g is a selfadjoint operator in the Hilbert space $L^2(M)$. Its resolvent

$$(1) \quad R_z(\Delta_g) = (\Delta_g - z)^{-1}$$

is a holomorphic function on $\mathbb{C} \setminus \text{spec}(\Delta_g)$ with values in the bounded operators in $L^2(M)$. Whereas $\mathbb{C} \setminus \text{spec}(\Delta_g)$ is the largest domain where all matrix elements of the resolvent

$$\langle \phi, R_z(\Delta_g)\psi \rangle \quad \phi, \psi \in L^2(M)$$

are holomorphic functions it may happen that for all ψ and ϕ in some dense subset of $L^2(M)$ those matrix elements can be continued to holomorphic functions on a Riemann surface extending the resolvent set $\mathbb{C} \setminus \text{spec}(\Delta_g)$.

A lot of examples where such a situation occurs are known. If (M, g) is the Euclidean space of dimension n , then the functions

$$\langle \phi, R_z(\Delta_g)\psi \rangle \quad \phi, \psi \in L^2(\mathbb{R}^n, e^{+\epsilon x^2} dx), \quad \epsilon > 0$$

extend to holomorphic functions on the concrete Riemann surface associated with either the function \sqrt{z} or $\log(z)$, depending on whether the dimension n is odd or even (see e.g. [18]). If a compactly supported potential is added, the same holds, except that in

this case the continuation is meromorphic. Poles of these matrix elements are commonly referred to as scattering poles or resonances. The theory that provides analytic continuation in the case of the Laplacian in \mathbb{R}^n with potential is well developed and plays a crucial role for example in proving the absence of singular continuous spectrum (see e.g. [4] and references therein). Also in order to make sense of the notion of resonances one needs to proof the existence of a meromorphic continuation across the spectrum. From the results known for the Euclidean space one may obtain further examples employing perturbation techniques (e.g. [2]).

Here we will be mainly interested in a continuation of the resolvent kernel, i.e. we view $R_z(\Delta_g)$ as a function with values in $\mathcal{D}'(M \times M)$. Hence, we look at the matrix elements $\langle \phi, R_z(\Delta_g)\psi \rangle$ where $\phi, \psi \in C_0^\infty(M)$.

Examples of manifolds where a meromorphic continuation of the resolvent kernel is known to exist include manifolds with cylindrical ends ([18, 20, 17]), certain strictly pseudoconvex domains ([5]), manifolds with asymptotically constant negative curvature ([16, 8]), and noncompact symmetric spaces of rank 1 ([19, 3]). Concerning this question little was known however about higher rank symmetric spaces of noncompact type. Only recently the method of complex scaling was used by Mazzeo and Vasy ([14, 15]) to show that the resolvent kernel has a meromorphic extension across the spectrum to a region which is obtained by rotating the spectrum up to the angle $\pi/2$ and by adding further branching points. It was conjectured however in their papers that this continuation has no poles and that the surface can be chosen without branching points. A special case which can be considered as well understood is that of a symmetric space $X = G/K$, where G is a complex semisimple Lie group. In this case there is an explicit formula for the heat kernel (see [6]). By Laplace transform one can obtain an explicit expression for the resolvent kernel in terms of the modified Bessel function of the second kind. In this way one finds that the resolvent kernel admits a holomorphic extension to a branched cover of the complex plane.

We will show that in the case of a noncompact Riemannian symmetric space $X = G/K$ of arbitrary rank the resolvent kernel admits a holomorphic extension to a Riemann surface depending on the rank of the symmetric space. This Riemann surface is a branched cover of the complex plane with a part of the real axis removed. A branching point occurs at the bottom of the spectrum. This branching point turns out to be quadratic if the rank of the space is odd and logarithmic otherwise. This Riemann surface contains the region obtained by rotating the spectrum up to the angle π , and our result implies the absence of poles in that region. In case the rank of the space is one we reproduce the known results, i.e. the resolvent kernel extends to a meromorphic function on the branched cover of \mathbb{C} associated with the function $\sqrt{z - \mu}$, where μ is the

bottom of the spectrum. We get even stronger results in case the group G has only one conjugacy class of Cartan subalgebras. In this case the resolvent kernel behaves almost like in Euclidean space, i.e. there exists a holomorphic continuation to a Riemann surface which is a branched cover of \mathbb{C} with the only branching point being the bottom of the spectrum. The branching point is quadratic or logarithmic depending on whether the rank of the space is odd or even. This special case includes the real hyperbolic spaces of odd dimension, the spaces $SU^*(2n)/Sp(n)$, $E_{6(-26)}/F_4$ as well as the case G/K when G is a complex group.

Our method relies on the Fourier transform on symmetric spaces of noncompact type and on the meromorphic continuation of the Harish-Chandra c -function.

2. NOTATIONS AND BACKGROUND MATERIAL

2.1. Symmetric spaces of noncompact type. Suppose that $X = G/K$ is a symmetric space of noncompact type, that is G is a real connected noncompact semisimple Lie group with finite center and $K \subset G$ is a maximal compact subgroup. Let \mathfrak{g} be the Lie algebra of G and let $\mathfrak{k} \subset \mathfrak{g}$ be the Lie algebra of K . We denote by $\kappa(\cdot, \cdot)$ the Killing form on \mathfrak{g} , i.e.

$$(2) \quad \kappa(X, Y) = \text{Tr}(\text{Ad}(X) \circ \text{Ad}(Y)).$$

Then there exists a Cartan involution $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$ with fixed point algebra \mathfrak{k} , i.e. θ is an involutive automorphism such that the bilinear form $\langle X, Y \rangle := -\kappa(X, \theta Y)$ is positive definite and such that the $+1$ eigenspace of θ coincides with \mathfrak{k} . Hence, the decomposition of \mathfrak{g} into $+1$ and -1 eigenspaces of θ reads

$$(3) \quad \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{p} \subset \mathfrak{g}$ is a linear subspace. If $\mathfrak{g}_{\mathbb{C}}$ is the complexification of \mathfrak{g} then

$$(4) \quad \mathfrak{g}_c = \mathfrak{k} + i\mathfrak{p} \subset \mathfrak{g}_{\mathbb{C}}$$

is a compact real form of $\mathfrak{g}_{\mathbb{C}}$. Now let \mathfrak{a} be a maximal abelian subspace of \mathfrak{p} and let \mathfrak{m} be the centralizer of \mathfrak{a} in \mathfrak{k} . If \mathfrak{b} is a maximal abelian subalgebra of \mathfrak{m} then $\mathfrak{h} = \mathfrak{b} + \mathfrak{a}$ is a θ -stable Cartan subalgebra of \mathfrak{g} . The set of restricted roots $\Delta(\mathfrak{g}, \mathfrak{a})$ w.r.t. \mathfrak{a} is the set of nonzero linear functionals $\alpha \in \mathfrak{a}^*$ such that

$$(5) \quad [x, y] = \alpha(x)y, \quad \forall x \in \mathfrak{a}$$

for some nonzero element $y \in \mathfrak{g}$. The multiplicity m_{α} of a restricted root is the dimension of the vector space $\{y \in \mathfrak{g}; [x, y] = \alpha(x)y, \quad \forall x \in \mathfrak{a}\}$. This gives rise to a root space

decomposition of \mathfrak{g} :

$$(6) \quad \mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \sum_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})} \mathfrak{g}_\alpha,$$

where \mathfrak{g}_α are the root subspaces, i.e.

$$(7) \quad \mathfrak{g}_\alpha = \{y \in \mathfrak{g}; [x, y] = \alpha(x)y, \quad \forall x \in \mathfrak{a}\}.$$

Note that each restricted root $\Delta(\mathfrak{g}, \mathfrak{a})$ coincides with the restriction of a root $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$ to \mathfrak{a} . We may choose now a subsystem of positive roots $\Delta^+(\mathfrak{g}, \mathfrak{h})$ and accordingly a subsystem of positive restricted roots $\Delta^+(\mathfrak{g}, \mathfrak{a})$. As usual we use the notation

$$(8) \quad \rho = \frac{1}{2} \sum_{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})} m_\alpha \alpha$$

for half of the sum of positive roots counting multiplicities. Let $\mathfrak{n} \subset \mathfrak{g}$ be the nilpotent Lie-subalgebra

$$(9) \quad \mathfrak{n} = \sum_{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})} g_\alpha.$$

Then the Iwasawa decomposition of \mathfrak{g} reads as follows:

$$(10) \quad \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

Now let A and N be the analytic subgroups of G with Lie algebras \mathfrak{a} and \mathfrak{n} respectively. Then

$$(11) \quad N \times A \times K \rightarrow G, (n, a, k) \rightarrow nak$$

is a diffeomorphism from $N \times A \times K$ to G . This is the classical Iwasawa decomposition of G . For $g \in G$ we denote by $A(g) \in \mathfrak{a}$ the unique element such that g can be expressed as

$$(12) \quad g = n \exp(A(g))k$$

with $n \in N$ and $k \in K$. Let M be the centralizer of A in K and denote by B be the compact homogeneous space K/M . Then $A(gK, kM) := A(k^{-1}g)$ defines a smooth function on $X \times B$ with values in \mathfrak{a} . If M' is the normalizer of A in K then the restricted Weyl group is defined by $W := M'/M$. The restricted Weyl group acts on \mathfrak{a} by $kMa = \text{Ad}(k)a$ and by duality it also acts on the dual \mathfrak{a}^* of \mathfrak{a} . This representation of W on \mathfrak{a} is injective and W can be identified in this way with a group of reflections in \mathfrak{a} . As usual let \mathfrak{a}_+^* be the positive Weyl chamber $\{\lambda \in \mathfrak{a}^*; \lambda(\alpha) > 0 \quad \forall \alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})\}$. Then \mathfrak{a}_+^* is a fundamental domain for the action of W on \mathfrak{a}^* .

2.2. The Helgason transform and the Paley-Wiener theorem. Let $X = G/K$ be a symmetric space of noncompact type. We use the notations and conventions from above. The generalized Fourier transform (Helgason transform) of a function $f \in C_0^\infty(X)$ is the function $\hat{f} : \mathfrak{a}^* \times B \rightarrow \mathbb{C}$ defined by

$$(13) \quad \hat{f}(\lambda, b) = \int_X f(x) e^{(-i\lambda + \rho)(A(x, b))} dx,$$

where we integrate with respect to some invariant measure on X which we choose to be normalized as in [10]¹. For suitable normalizations of the Euclidean measure on \mathfrak{a}^* and the invariant measure db on B the inverse Fourier transform (see [10]) is given by

$$(14) \quad f(x) = w^{-1} \int_{\mathfrak{a}^* \times B} \hat{f}(\lambda, b) e^{(+i\lambda + \rho)(A(x, b))} \frac{d\lambda}{|\mathbf{c}(\lambda)|^2} db,$$

where $\mathbf{c}(\lambda)$ is the Harish-Chandra \mathbf{c} -function, and w is the order of the restricted Weyl group. Moreover, for $f, g \in C_0^\infty(X)$ we have

$$(15) \quad \int_X \bar{f}(x) g(x) dx = w^{-1} \int_{\mathfrak{a}^* \times B} \bar{\hat{f}}(\lambda, b) \hat{g}(\lambda, b) \frac{d\lambda}{|\mathbf{c}(\lambda)|^2} db.$$

The Fourier transform \mathcal{F} extends to an isometry

$$L^2(X) \rightarrow L^2(\mathfrak{a}_+^* \times B, \frac{d\lambda}{|\mathbf{c}(\lambda)|^2} db).$$

For $f \in C_0^\infty(X)$ the Fourier transform \hat{f} extends to a function on $\mathfrak{a}_+^* \times B$ which is entire in the first variable. Let $\mathcal{H}(a_+^*)$ be the space of holomorphic functions of uniform exponential type on $a_+^* \times B$, i.e. the space functions ϕ on $a_+^* \times B$, entire in the first variable, such that there exists a constant $R > 0$ with

$$(16) \quad |\phi(\lambda, b)| \leq C_N (1 + |\lambda|_\kappa)^{-N} e^{R|\Im(\lambda)|_\kappa}$$

for all $N \in \mathbb{N}_0$ with some constants $C_N > 0$. Here $|\cdot|_\kappa$ denotes the norm induced by the Killing form. The Paley-Wiener theorem for symmetric spaces of noncompact type (see [11]) is precisely the statement that the image of $C_0^\infty(X)$ under the Fourier transform coincides with the space of functions $f \in \mathcal{H}(a_+^*)$ which satisfy

$$(17) \quad \int_B f(s\lambda, b) e^{(+is\lambda + \rho)(A(x, b))} db = \int_B f(\lambda, b) e^{(+i\lambda + \rho)(A(x, b))} db \quad \forall s \in W, x \in X, \lambda \in \mathfrak{a}_+^*.$$

The \mathbf{c} -Function satisfies $|\mathbf{c}(\lambda)|^2 = \mathbf{c}(\lambda)\mathbf{c}(-\lambda)$ and is known to extend to a meromorphic function on \mathfrak{a}_+^* . This follows from the product formula of Gindikin and Karpelevic ([7]) which leads to the following explicit expression for the \mathbf{c} -function.

$$(18) \quad \mathbf{c}(\lambda) = \mathbf{c}_0 \prod_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})_0^+} \frac{2^{-\langle i\lambda, \alpha_0 \rangle} \Gamma(\langle i\lambda, \alpha_0 \rangle)}{\Gamma(\frac{1}{2}(\frac{1}{2}m_\alpha + 1 + \langle i\lambda, \alpha_0 \rangle)) \Gamma(\frac{1}{2}(\frac{1}{2}m_\alpha + m_{2\alpha} + \langle i\lambda, \alpha_0 \rangle))},$$

¹the normalization of the measure is not important for our considerations

where $\alpha_0 = \frac{\alpha}{\langle \alpha, \alpha \rangle}$ and the constant \mathbf{c}_0 is determined by $\mathbf{c}(-i\rho) = 1$. The product is over the set $\Delta(\mathfrak{g}, \mathfrak{a})_0^+$ of indecomposable positive restricted roots, i.e.

$$(19) \quad \Delta(\mathfrak{g}, \mathfrak{a})_0 := \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})^+; \frac{1}{2}\alpha \notin \Delta(\mathfrak{g}, \mathfrak{a})\}.$$

The scalar product $\langle \cdot, \cdot \rangle$ is the positive definite scalar product on \mathfrak{a}^* induced by the Killing form.

2.3. The metric Laplacian. As above let $X = G/K$ be a symmetric space of non-compact type and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the corresponding Cartan decomposition. The tangent space $T_{eK}X$ at the point eK in X is canonically identified with \mathfrak{p} and invariant metrics on X are therefore in 1 – 1 correspondence with $Ad(K)$ -invariant positive definite scalar products on \mathfrak{p} . Suppose now that g is an invariant metric on X and denote by (\cdot, \cdot) the corresponding scalar product on \mathfrak{p} . This induces a scalar product on \mathfrak{a} and on its dual \mathfrak{a}^* . We denote the dual scalar product on \mathfrak{a}^* again by (\cdot, \cdot) ². The Laplace operator $\Delta_g : C_0^\infty(X) \rightarrow L^2(X)$ associated with the metric is a G -invariant differential operator and hence, its spectral decomposition is strongly related to the Fourier transform. For $\lambda \in \mathfrak{a}^*$ and $b \in B$ we define the function $\phi_{\lambda, b}$ on X by $\phi_{\lambda, b}(x) = e^{(-i\lambda + \rho)(A(x, b))}$ and obtain (see [9] and [11] p. 458, 462)³

$$(20) \quad \Delta_g \phi_{\lambda, b} = (|\lambda|^2 + |\rho|^2) \phi_{\lambda, b},$$

where the norm $|\lambda| = (\lambda, \lambda)^{1/2}$ is with respect to the scalar product induced by the metric g . As a consequence if $\mathcal{F} : L^2(X) \rightarrow L^2(\mathfrak{a}_+^* \times B, \frac{d\lambda}{|\mathbf{c}(\lambda)|^2} db)$ is the Fourier transform the operator $\mathcal{F} \Delta_g \mathcal{F}^{-1}$ is a multiplication operator. More precisely, if $m_{|\lambda|^2 + |\rho|^2}$ is the selfadjoint operator in $L^2(\mathfrak{a}_+^* \times B, \frac{d\lambda}{|\mathbf{c}(\lambda)|^2} db)$ defined by

$$(21) \quad (m_{|\lambda|^2 + |\rho|^2} f)(\lambda, k) = (|\lambda|^2 + |\rho|^2) f(\lambda, k).$$

with maximal domain, the operator $\mathcal{F} \Delta_g \mathcal{F}^{-1}$ coincides with its restriction to the set $\mathcal{F}(C_0^\infty(X))$. Therefore, $m_{|\lambda|^2 + |\rho|^2}$ coincides with $\mathcal{F} \overline{\Delta_g} \mathcal{F}^{-1}$, where $\overline{\Delta_g}$ denotes the closure of Δ_g in $L^2(X)$. Hence, the Fourier transform yields the spectral decomposition of the selfadjoint operator $\overline{\Delta_g}$.

²in contrast to the scalar product induced by the Killing form which we denoted by $\langle \cdot, \cdot \rangle$

³this formula appears in the literature for the case that the metric is induced by the Killing form. On each irreducible factor of the symmetric space any invariant metric is proportional to the one induced by the Killing form. Taking this into account it is easy to see that the formula holds in the general situation.

2.4. The resolvent of the Laplacian. Let \mathcal{H} be a Hilbert space and let A be a closed operator in \mathcal{H} . The resolvent set of A is defined to be the set of all $z \in \mathbb{C}$ such that the operator $A - z$ has a bounded inverse. The resolvent $R_z(A) := (A - z)^{-1}$ is a holomorphic function with values in $\mathcal{B}(\mathcal{H})$ on the resolvent set $\rho(A)$ of A . The complement of the resolvent set is called spectrum of A . If A is selfadjoint the spectrum is contained in the real line.

We want to investigate the resolvent of the closure of the Laplace operator Δ_g on a symmetric space $X = G/K$ of noncompact type with invariant Riemannian metric g . It follows from (21) and the above that the spectrum of $\overline{\Delta}_g$ is exactly the set

$$(22) \quad \text{spec}(\overline{\Delta}_g) = [|\rho|^2, \infty).$$

For $f, g \in L^2(X)$ we have by the spectral theorem

$$(23) \quad \begin{aligned} \langle f, (\Delta_g - z)^{-1}g \rangle &= \int_{\mathfrak{a}_+^* \times B} \frac{\overline{\hat{f}(\lambda, b)} \hat{g}(\lambda, b)}{|\lambda|^2 + |\rho|^2 - z} \frac{d\lambda}{|\mathbf{c}(\lambda)|^2} db = \\ &= w^{-1} \int_{\mathfrak{a}^* \times B} \frac{\overline{\hat{f}(\lambda, b)} \hat{g}(\lambda, b)}{|\lambda|^2 + |\rho|^2 - z} \frac{d\lambda}{|\mathbf{c}(\lambda)|^2} db. \end{aligned}$$

3. ANALYTIC CONTINUATION OF THE RESOLVENT KERNEL

Let $X = G/K$ by a symmetric space of noncompact type with invariant Riemannian metric g . Then the resolvent of the Laplacian

$$(24) \quad R_z(\overline{\Delta}_g) = (\overline{\Delta}_g - z)^{-1}$$

can be regarded as a map $C_0^\infty(X) \rightarrow \mathcal{D}'(X)$ and hence has a distributional kernel. This resolvent kernel is a holomorphic function on the set $\mathbb{C} \setminus [|\rho|^2, \infty)$ with values in $\mathcal{D}'(X \times X)$. Throughout this section r will denote the positive real number

$$(25) \quad r := \min\{|\alpha| j(m_\alpha)\}_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})_0^+},$$

where $j(x) = \frac{x}{2}$ if x is odd and $j(x) = \frac{x}{2} + 1$ if x is even. Note that here $|\alpha|$ is the length of α with respect to the metric g which is not necessarily induced by the Killing form. Our main result is that the resolvent kernel admits an analytic continuation to a larger Riemann surface.

Theorem 3.1. *Let $X = G/K$ be a symmetric space of noncompact type with invariant Riemannian metric g . Suppose that the rank of X is odd. Let $H^- := \{z \in \mathbb{C}; \Im(z) < 0\}$ be the lower half plane. For each $f, g \in C_0^\infty(X)$ define the holomorphic function*

$$F : H^- \rightarrow \mathbb{C}, \quad z \mapsto \langle f, (\Delta_g - |\rho|^2 - z^2)^{-1} g \rangle.$$

Then the function F admits a holomorphic continuation to the open set $\mathbb{C} \setminus i[r, \infty)$. If $\text{rk}(X) = 1$ the function F admits a meromorphic continuation to \mathbb{C} with all poles contained in the set $i[r, \infty)$.

Denote by Λ^o the concrete Riemann surface associated with the function $\sqrt{z - |\rho|^2}$ on the domain $\mathbb{C} \setminus [|\rho|^2, \infty)$. This means that Λ^o is a branched double cover of the complex plane with branching point $|\rho|^2$. The original domain $\mathbb{C} \setminus [|\rho|^2, \infty)$ is commonly referred to as the physical sheet, whereas its complement with the spectrum $[|\rho|^2, \infty)$ removed is called the unphysical sheet. Let Λ^o_r be the surface with the half line $(-\infty, |\rho|^2 - r^2]$ removed on the unphysical sheet. Then the above means that the resolvent kernel regarded as a distribution extends to a function which is holomorphic on Λ^o_r . In case the rank is 1 the resolvent kernel extends to a meromorphic function on Λ^o with all poles contained in the half line $(-\infty, |\rho|^2 - r^2]$ on the unphysical sheet. Note that in the literature sometimes another parameterization is chosen. By a change of variables $z = i(s - |\rho|)$ one can see that the functions

$$(26) \quad \langle f, (\Delta_g - s(4|\rho|^2 - s))^{-1} g \rangle,$$

which are defined in the half plane $\Re(s) < |\rho|$ have an analytic continuation across the line $\Re(s) = |\rho|$.

In case the symmetric space has even rank there is an analogous result.

Theorem 3.2. *Let $X = G/K$ by a symmetric space of noncompact type with invariant Riemannian metric g . Suppose that the rank of X is even. Denote by $S_{a,b}$ the strip $\{z \in \mathbb{C}; a < \Im(z) < b\}$. Then for each $f, g \in C_0^\infty(X)$ the holomorphic function*

$$F : S_{-\pi, 0} \rightarrow \mathbb{C}, \quad z \mapsto \langle f, (\Delta_g - |\rho|^2 - e^{2z})^{-1} g \rangle$$

admits a holomorphic continuation to the open set

$$\mathcal{U} := \{z \in \mathbb{C}; z \notin i\pi(n + \frac{1}{2}) + [\log(r), \infty) \quad \forall n \in (\mathbb{Z} \setminus \{-1\})\}.$$

Let Λ^e be the concrete Riemann surface associated with the function $\log(z - |\rho|^2)$ on the domain $\mathbb{C} \setminus [|\rho|^2, \infty)$. This means, Λ^e is the logarithmic covering of $\mathbb{C} \setminus |\rho|^2$. Denote by Λ^e_r the Riemann surface Λ^e with the half line $(-\infty, |\rho|^2 - r^2]$ removed on all unphysical sheets. Then our result means that the resolvent kernel regarded as a distribution has a holomorphic extension to Λ^e_r .

If all Cartan subalgebras of G are conjugate there are even stronger statements. This condition is known to be equivalent to each of the following (see [12], Ch. IX, Th. 6.1)

- (1) all restricted roots have even multiplicity, i.e. m_α is even for all $\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})$.
- (2) $\text{rk}(G) = \text{rk}(X) + \text{rk}(K)$.

In this case it follows that $m_{2\alpha} = 0$ and that the function $\mathbf{c}(\lambda)^{-1}$ is a polynomial in λ (see [13], Ch. IV, Cor. 6.15). The irreducible symmetric spaces of noncompact type, where this happens are

- the real hyperbolic spaces of odd dimension, i.e. $SO_0(2n+1, 1)/SO(2n)$,
- the spaces $SU^*(2n)/Sp(n)$,
- the spaces G/K where G complex,
- the exceptional space $E_{6(-26)}/F_4$.

Theorem 3.3. *Let $X = G/K$ by a symmetric space of noncompact type with invariant Riemannian metric g . Suppose that G has only one conjugacy class of Cartan subalgebras. Let $f, g \in C_0^\infty(X)$ and*

$$F(z) := \langle f, (\Delta_g - |\rho|^2 - z)^{-1}g \rangle$$

Then the following holds.

- (1) *if $\text{rk}(X)$ is odd then $F(z^2)$ has an analytic continuation to the whole complex plane.*
- (2) *if $\text{rk}(X)$ is even then $F(e^{2z})$ has an analytic continuation to the whole complex plane.*

Hence, in this special situation the resolvent kernel has a holomorphic extension to Λ^e or Λ^o , depending on whether the rank of X is even or odd.

4. PROOF OF THE MAIN RESULTS

We will split the proof of this theorem into several propositions. By the spectral theorem we have for $f, g \in C_0^\infty(X)$ and $z \notin [|\rho|^2, \infty)$

$$\begin{aligned} (27) \quad & \langle f, (\Delta_g - z)^{-1}g \rangle = \\ & = w^{-1} \int_{\mathfrak{a}^*} \frac{1}{|\mathbf{c}(\lambda)|^2(|\lambda|^2 + |\rho|^2 - z)} \int_B \overline{\hat{f}(\lambda, b)} \hat{g}(\lambda, b) db d\lambda = \\ & = \int_{\mathfrak{a}^*} \frac{1}{|\mathbf{c}(\lambda)|^2(|\lambda|^2 + |\rho|^2 - z)} V(\lambda) d\lambda, \end{aligned}$$

where $V(\lambda) = w^{-1} \int_B \hat{f}(-\lambda, b) \hat{g}(\lambda, b) db$ is rapidly decaying and admits a continuation to an entire function on \mathbb{C} . This follows from the analytic properties of the Fourier transforms and the fact that B is compact. Now denote by S the unit sphere in \mathfrak{a}^* , i.e. $S = \{\lambda \in \mathfrak{a}^*; |\lambda| = 1\}$. Then using polar coordinates we obtain

$$(28) \quad \langle f, (\Delta_g - |\rho|^2 - z)^{-1}g \rangle = \int_{\mathbb{R}^+} \frac{F(x)}{(x^2 - z)} dx,$$

where

$$(29) \quad F(x) = Cx^{\dim(\mathfrak{a}-1)} \int_S \frac{V(x\lambda)}{\mathbf{c}(x\lambda)\mathbf{c}(-x\lambda)} d\mu_S(\lambda),$$

μ_S is the usual measure on the sphere and C is a constant not depending on x .

Proposition 4.1. *The function $F : \mathbb{R}^+ \rightarrow \mathbb{C}$ defined by (29) is bounded and admits a holomorphic continuation to the set $\mathbb{C} \setminus (\mathbf{i}[r, \infty) \cup -\mathbf{i}[r, \infty))$ (see Fig. 1). Moreover, $F(-z) = (-1)^{\text{rk}(X)-1}F(z)$ and $\lim_{z \rightarrow 0} z^{1-\text{rk}(X)}F(z) = 0$. In case m_α is even for all $\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})_0^+$ the function F admits a continuation to an entire function. In case the rank of X is one then F has a meromorphic extension to the whole complex plane.*

Proof. The inverse of the \mathbf{c} -function is polynomially bounded (see [13], Ch. IV, Prop. 7.2) whereas V is rapidly decreasing. This immediately implies that F is rapidly decreasing as well and hence, bounded. For each $\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})_0^+$ we define the meromorphic function

$$(30) \quad h_\alpha(z) := \frac{\Gamma(\frac{1}{2}(\frac{1}{2}m_\alpha + 1 + z))\Gamma(\frac{1}{2}(\frac{1}{2}m_\alpha + m_{2\alpha} + z))}{\Gamma(z)}.$$

Note that all poles of h_α are on the negative real axis. In case m_α is even and $m_{2\alpha} = 0$ the function h_α is entire in z . If m_α is odd it is known that $m_{2\alpha} = 0$ (see [1], or [12] Chapter X, Exercise F). Hence, in this case the set of poles of h_α is $-(\frac{m_\alpha}{2} + n) \quad n \in \mathbb{N}_0$. As in (25) let $j(x) = \frac{x}{2}$ if x is odd and $j(x) = \frac{x}{2} + 1$ if x is even. Hence, if $|z| < j(m_\alpha)$ then z is not a pole of h_α . Now suppose that $|\lambda| < |\alpha|j(m_\alpha)$. Then we have

$$(31) \quad |\langle \lambda, \frac{\alpha}{\langle \alpha, \alpha \rangle} \rangle| \leq |\lambda|_\kappa |\alpha|_\kappa^{-1} = |\lambda| |\alpha|^{-1} < j(m_\alpha),$$

and therefore the function $h_\alpha(\langle i\lambda, \alpha_0 \rangle)$ is analytic in the ball $|\lambda| < r$, where r is defined by (25). By the product formula (18) the function $\frac{1}{\mathbf{c}(-\lambda)\mathbf{c}(\lambda)}$ is analytic in the ball $|\lambda| < r$. Now suppose that $z \in \mathbb{C} \setminus i\mathbb{R}$ and $\lambda \in S$. Then we have either $\langle iz\lambda, \alpha_0 \rangle = 0$ or $\Im(\langle iz\lambda, \alpha_0 \rangle) \neq 0$. In this case $z\lambda$ is not a pole of $h_\alpha(\langle i\lambda, \alpha_0 \rangle)$. We conclude that the function $\mathbb{C} \setminus (i[r, \infty) \cup -i[r, \infty)) \times S \rightarrow \mathbb{C}$

$$(z, \lambda) \rightarrow \frac{V(z\lambda)}{\mathbf{c}(z\lambda)\mathbf{c}(-z\lambda)}$$

is holomorphic in the first variable. Hence, the defining formula (29) for the function F yields the desired analytic continuation. The formula for $F(-z)$ is immediate from (29). Since the Γ -function has a pole at 0 the function $\mathbf{c}(\lambda)^{-1}$ vanishes at 0. Hence, the integral on the right hand side of (29) vanishes at $\lambda = 0$. This implies $\lim_{z \rightarrow 0} z^{1-\text{rk}(X)}F(z) = 0$. If all m_α are even the function $\frac{1}{\mathbf{c}(-\lambda)\mathbf{c}(\lambda)}$ is a polynomial and hence entire. In this case F is entire as well by the same argument. In case the rank of X is one the function F is meromorphic in the whole complex plane, since in this case the function $\frac{1}{\mathbf{c}(-\lambda)\mathbf{c}(\lambda)}$ is a meromorphic function of one variable and the integral in (29) is a sum. \square

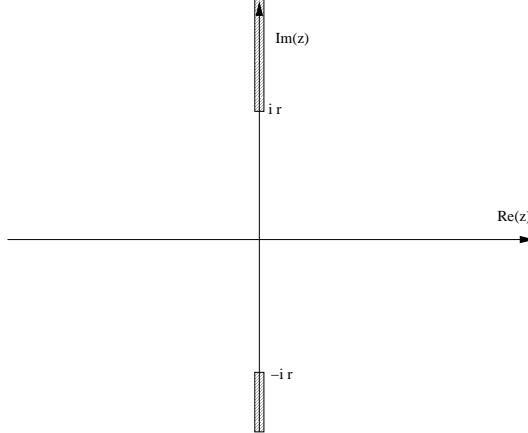


FIGURE 1. Domain of analyticity of the function $F(z)$

By the above observation we are interested in evaluating integrals of the form

$$\int_{\mathbb{R}^+} \frac{f(x)}{x^2 - z} dx,$$

where f is a meromorphic function on $\mathbb{C} \setminus (i[r, \infty) \cup -i[r, \infty))$ with no poles on the real axis. This integral clearly defines a holomorphic function on $\mathbb{C} \setminus \mathbb{R}^+$, and we may ask, whether it has an extension to a meromorphic function on a larger Riemann surface. We have the following results.

Proposition 4.2. *Let r be a positive real number and suppose that f is a meromorphic function on the set $\mathbb{C} \setminus (i[r, \infty) \cup -i[r, \infty))$ such that*

$$|f(x)| \leq C, \quad \forall x \in \mathbb{R} \quad \text{and}$$

$$f(x) = f(-x), \quad \forall x \in \mathbb{R}.$$

Define the holomorphic function G on $H^- := \{z \in \mathbb{C}; \Im(z) < 0\}$ by

$$G(z) := \int_{\mathbb{R}} \frac{f(x)}{x^2 - z^2} dx.$$

Then $G(z)$ has a meromorphic continuation to $\mathbb{C} \setminus i[r, \infty)$. Except for the point 0 all poles of $G(z)$ are contained in the set of poles of f . The singular behaviour of $G(z)$ at 0 is like $\frac{i\pi f(z)}{z}$. Hence, if $f(0) = 0$ then 0 is not a pole.

Proof. The function G is clearly holomorphic in the lower half plane, since the integral converges absolutely. Let γ be a path in \mathbb{C} chosen like in Fig. 2, such that γ does not meet any poles. Denote the domain $\mathbb{C} \setminus (i[r, \infty) \cup -i[r, \infty))$ by \mathcal{U} . The path divides \mathcal{U} into two components. Let \mathcal{U}_γ^- be the interior of the lower component. Then

$$(32) \quad G_\gamma(z) := \int_{\gamma} \frac{f(x)}{x^2 - z^2} dx$$

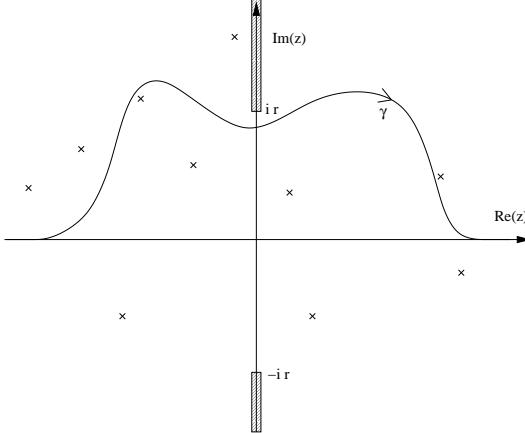


FIGURE 2. The path γ , poles of $f(z)$ are indicated by crosses

defines a holomorphic function on the open set $O_\gamma = \mathcal{U}_\gamma^- \cap -\mathcal{U}_\gamma^-$. Let H^+ and H^- be the upper and lower half plane respectively, i.e. $H^\pm = \{z \in \mathbb{C}; \Im(z) \gtrless 0\}$.

By Cauchies theorem we have for $z \in O_\gamma \cap H^-$

$$(33) \quad G(z) - G_\gamma(z) = 2\pi i \sum_a \text{Res}_{x=a} \left(\frac{f(x)}{x^2 - z^2} \right),$$

where the sum is taken over all poles a contained in $O_\gamma \cap H^+$ of the function $x \mapsto \frac{f(x)}{x^2 - z^2}$. Now choose $z \in O_\gamma \cap H^-$ such that $-z$ is not a pole of f . Then the function $\frac{f(x)}{x^2 - z^2}$ has a simple pole at $x = -z$ with residuum $\frac{f(z)}{2z}$. If a is a pole of f of order k we have

$$(34) \quad \text{Res}_{x=a} \left(\frac{f(x)}{x^2 - z^2} \right) = \frac{1}{(k-1)!} \left(\frac{\partial}{\partial x} \right)^{k-1} \frac{f(x)}{x^2 - z^2} \Big|_{x=a}$$

This expression is a finite sum of terms of the form $b_n(z^2 - a^2)^{-n}$ where the constants b_n do not depend on z . Hence, if a is a pole of f then $R_a(z) = \text{Res}_{x=a} \left(\frac{f(x)}{x^2 - z^2} \right)$ is a meromorphic function of z in the whole complex plane with poles only at $z = \pm a$. Let \mathcal{P} be the set of poles of f in $O_\gamma \cap H^+$. We conclude that

$$(35) \quad R(z) = \frac{f(z)}{2z} + \sum_{a \in \mathcal{P}} R_a(z)$$

extends to a meromorphic function on the complex plane with poles only at 0 and at the poles of f . The residuum of $R(z)$ at 0 is $f(0)/2$. Moreover, if $z \in O_\gamma \cap H^-$ and $-z$ is not a pole of f we have the equation

$$(36) \quad G(z) - G_\gamma(z) = 2\pi i R(z).$$

Since the set of such points is open this equation holds everywhere in $O_\gamma \cap H^-$. Since $G_\gamma(z)$ is holomorphic in O_γ the equation

$$(37) \quad G(z) = 2\pi i R(z) + G_\gamma(z)$$

defines a meromorphic continuation of $G(z)$ to $O_\gamma \cup H^-$. Since γ can be chosen such that an arbitrary point in $\mathbb{C} \setminus i[r, \infty)$ is contained in O_γ this completes the proof. \square

This allows us to analytically continue integrals of the form

$$\int_{\mathbb{R}^+} \frac{f(x)}{x^2 - z^2} dx$$

in case $f(x) = f(-x)$. The case $f(x) = -f(-x)$ is covered by the following proposition.

Proposition 4.3. *Suppose that $f : \mathbb{C} \setminus (i[r, \infty) \cup -i[r, \infty)) \rightarrow \mathbb{C}$ is a holomorphic function such that*

$$\begin{aligned} |f(x)| &\leq C, \quad \forall x \in \mathbb{R} \quad \text{and,} \\ f(x) &= -f(-x), \quad \forall x \in \mathbb{R}. \end{aligned}$$

Let $S_{a,b}$ be the strip $\{z \in \mathbb{C}; a < \Im(z) < b\}$ and define the holomorphic function G on $S_{-\pi,0}$ by

$$G(z) := \int_{\mathbb{R}^+} \frac{f(x)}{x^2 - e^{2z}} dx.$$

Denote by \mathcal{U} the open set

$$\mathcal{U} := \{z \in \mathbb{C}; z \notin i\pi(n + \frac{1}{2}) + [\log(r), \infty) \quad \forall n \in (\mathbb{Z} \setminus \{-1\})\}.$$

Then G has an analytic continuation to \mathcal{U} . Moreover, denoting the analytic continuation again by G we have

$$G(z + i\pi) - G(z) = \pi i f(e^z) e^{-z}.$$

Proof. A simple change of variables gives

$$(38) \quad G(z) = \int_{\mathbb{R}} \frac{f(e^x)}{e^{2x} - e^{2z}} e^x dx.$$

We define \mathcal{U}_0 by

$$\mathcal{U}_0 := \{z \in \mathbb{C}; z \notin i\pi(n + \frac{1}{2}) + [\log(r), \infty) \quad \forall n \in \mathbb{Z}\}.$$

By assumption the function $g(x) := f(e^x)e^x$ is holomorphic in \mathcal{U}_0 and is periodic in the sense that $g(x + \pi i) = g(x)$. Now choose a path γ like in Fig. 3. Denote by O_γ the connected component of $-\infty - i\frac{\pi}{2}$ in $\mathcal{U}_0 \setminus (\gamma \cup (\gamma - i\pi\mathbb{Z}))$. Then

$$(39) \quad G_\gamma(z) = \int_\gamma \frac{g(x)}{e^{2x} - e^{2z}} dx.$$

defines a function holomorphic in O_γ . Now let z be a point in $O_\gamma \cap S_{-\pi,0}$. Then by Cauchy's integral theorem we have

$$(40) \quad G(z) = G_\gamma(z).$$

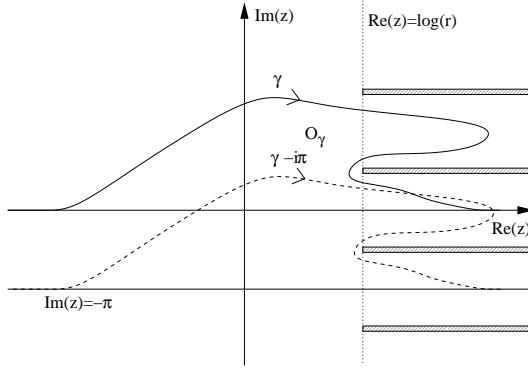


FIGURE 3. The Path γ

For each point $x \in \mathcal{U}_0 \cap H^+$ there is a curve, such that $x \in O_\gamma$. Hence, Equ. (40) defines an analytic continuation of G to $\mathcal{U} \cap H^+$. Suppose now that γ_1 and γ_2 are two different paths like in Fig. 3 such that $z \in O_{\gamma_1}$ and $z + i\pi \in O_{\gamma_2}$. Then Cauchy's integral theorem gives

$$(41) \quad G_{\gamma_2}(z + i\pi) - G_{\gamma_1}(z) = 2\pi i \frac{g(z)}{2} e^{-2z} = \pi i f(e^z) e^{-z}.$$

Hence, the analytically continued function satisfies

$$(42) \quad G(z + i\pi) - G(z) = \pi i f(e^z) e^{-z}.$$

This formula provides an extension of G to all of \mathcal{U} satisfying (42). \square

In view of the formula (28) the combination of Prop. 4.1 with Prop. 4.2 and Prop. 4.3 proves the Theorems 3.1, 3.2 and 3.3.

5. CONCLUDING REMARKS

By elliptic regularity the resolvent kernel (and its analytic continuation) is smooth off the diagonal. Hence, we automatically obtain an analytic continuation of the functions $R_z(x, y)$ for $x \neq y$ to the same Riemann surfaces.

Our result may be used to obtain meromorphic continuations of resolvent kernels associated to operators of the form $\Delta_g + V$, where V is a compactly supported potential. This can be obtained using standard perturbation arguments based on the analytic Fredholm theorem.

We would also like to mention that our method does not rely on the fact that the Fourier transforms of the test functions are in the space of rapidly decaying functions of uniform exponential type $\mathcal{H}(a_C^*)$. We only use the fact that the Fourier transforms extend to entire functions on a_C^* and that the function F defined by (29) is integrable. The latter happens to be the case for all $f, g \in L^2(X)$. The former requires certain growth conditions for f and g at infinity. Suppose for example that α is a smooth

function on X such that $\alpha > 1$ and such that for all $b \in B$ the functions $\alpha(x)^{-\frac{1}{2}}e^{\lambda(A(x,b))}$ are entire functions of λ with values in $L^2(X)$. Then our results remain true for f, g in the weighted Hilbert space $L^2(X, \alpha(x)dx)$. Denoting $\mathcal{H}_+ = L^2(X, \alpha(x)dx)$ and $\mathcal{H}_- = L^2(X, \alpha^{-1}(x)dx)$ we therefore see that the resolvent regarded as function with values in $\mathcal{L}(\mathcal{H}_+, \mathcal{H}_-)$ has an analytic (meromorphic) continuation to the considered Riemann surfaces. This may be useful when considering perturbations of the Laplace operator by potentials that are not compactly supported.

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